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RADIATION PATTERNS FROM A VIBRATING PLATE

by E. Bromberg & H. Cohn

A Memorandum Submitted by the
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INTRODUCTION

The note submitted herewith deals with the control of the directivity pattern of sound radiated from a plate vibrating in an infinite baffle. It supplements recent investigations of the problem such as those by the Harvard Underwater Sound Laboratory and by H. Nyquist* in so far as complete mathematical solution of the following problem is given: How can a prescribed radiation pattern be obtained by a suitable distribution of amplitudes (and phases) over a vibrating plate? The general answer, obtained for circular plates in an explicit form, can, of course, be realized in practice only to a more or less approximate degree. Nevertheless, the idealized solution given herein will serve as an indication of how the designing engineer could profitably distribute amplitudes (and phases) over the various elements of a mosaic of vibrating elements which together form the whole of a projector.

R. Courant

Official Investigator

* "Report on Directivity Pattern of Sound Sources" O.S.R.D. No. 706
"Directivity of Sound in Water - Elementary Arrays" C4 - NDRC - 072

RADIATION PATTERNS FROM A VIBRATING PLATE

by E. Bromberg and H. Cohn

The problem of the control of the directivity pattern of sound radiated from a vibrator set in an infinite rigid wall or "baffle" has recently* been treated by the Harvard Underwater Sound Laboratory and by Dr. Harry Nyquist of Division C Section 4. Their vibrators consisted of line sources and point sources.**

We wish to consider the same problem for a two-dimensional array of sound sources so densely packed that it may be considered as a continuous surface distribution of a vibrating plate. We assume further that the overall dimensions of the plate and the frequency of vibration have been specified.

We take our plate in the x-y plane vibrating in the $\pm z$-direction, so that the equation of the surface of the plate at any time t is

$$z = \mathrm{Re}\left\{ f(x,y)\ e^{-i\omega t} \right\} \tag{1}$$

where $f(x,y)$ is a given amplitude function, extending over the projection of the plate on the x-y plane. If all points on the plate vibrate in phase (or out of phase by 180°) $f(x,y)$ is real and positive (or negative); otherwise $f(x,y)$ is complex. We shall study the dependence of the radiation pattern on the amplitude function $f(x,y)$.

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* "Report on Directivity Pattern of Sound Sources" O.S.R.D. No. 706
"Directivity of Sound in Water - Elementary Arrays" C4 - N.D.R.C. - 072

** Further investigation of the regulation of line sources is being made by Mr. R. P. Shaw of the N. Y. U. Applied Mathematics Group.

The complex amplitude function of the acoustic pressure, at the point of observation, $P(x_o, y_o, z_o)$, is given by:

$$P = \frac{\rho\omega^2}{2\pi}\int_{\text{plate}}\int \frac{e^{ikr}}{r} f(x,y)\,dx\,dy \tag{2}$$

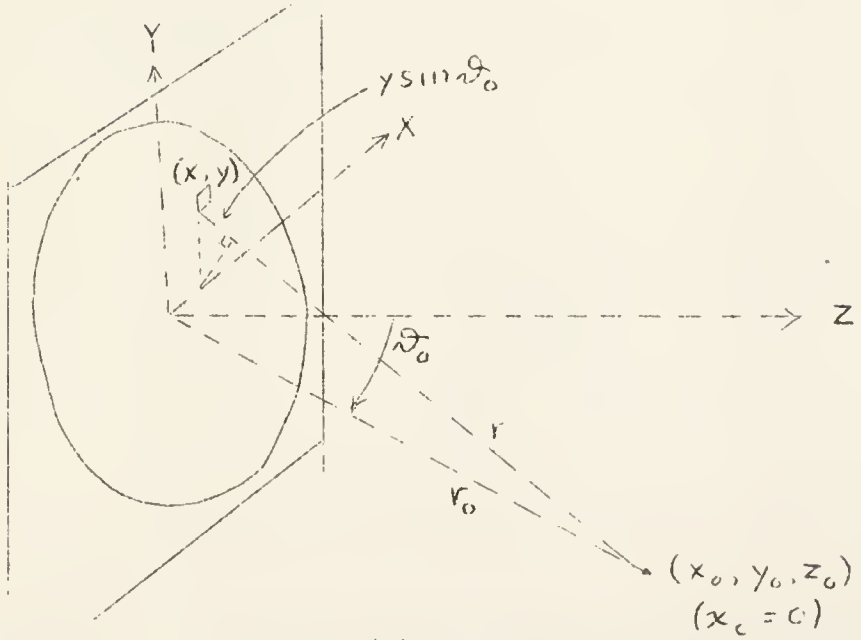
where

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z_o)^2}$$

In what follows we confine our attention to a circular plate where

$$f(x,y) = f(R), \qquad R = \sqrt{x^2 + y^2} \qquad 0 < R < A,$$

where A is the radius of the plate.



We now consider formula (a) at large distances from the plate. Then

$$r \approx r_o + y \sin\vartheta_o$$

where $r_o = \sqrt{x_o^2 + y_o^2 + z_o^2}$, ϑ_o is the angle between the radial vector to (x_o, y_o, z_o) and the z axis, and the x,y axes are chosen, for convenience, so that $x_o = 0$. Hence (2) may be simplified:

$$P \approx \frac{\rho\omega^2}{2\pi} e^{ikr_0} \iint_{x^2+y^2<A^2} e^{iky\sin\vartheta_0} f(\sqrt{x^2+y^2})\, dx\, dy$$

$$\approx \rho\omega^2 \frac{e^{ikr_0}}{r_0} g(k\sin\vartheta_0)$$

where we define g by

$$g(\mu) = \frac{1}{2\pi} \iint_{x^2+y^2<A^2} e^{i\mu y} f(\sqrt{x^2+y^2})\, dx\, dy$$

$$= \frac{1}{2\pi} \int_{R=0}^{A} f(R)\, R\, dR \int_0^{2\pi} e^{i\mu R\sin\varphi}\, d\varphi$$

$$(3) \quad \therefore\ g(\mu) = \int_{R=0}^{A} R\, f(R)\, J_0(\mu R)\, dR$$

The complex amplitude of the wave is then given by $g(k\sin\vartheta_0)\frac{e^{ikr_0}}{r_0}$ = const. and its intensity by const. $\times \left| \frac{g(k\sin\vartheta_0)}{r_0} \right|^2$. The "amplitude radiation pattern" is defined by

$$\gamma(\mu) = |g(k\sin\vartheta)|$$

If the plate is rigid, that is, $f(R)$ = constant $= v_1$, we have the well known result

$$(4) \qquad g(\mu) = v_1 \, A^2 \, \frac{J_1(A\mu)}{A\mu}$$

Let us now suppose that the radiation pattern is preassigned, i.e., the non-negative function $\gamma(\vartheta)$ is given for $0 < \vartheta < \frac{\pi}{2}$. This defines $g(\vartheta)$ for $0 < \mu < k$ (g may be taken as zero elsewhere). Then if we find a "mode of vibration" $f(R)$ such that

$$(3a) \qquad g(\mu) = \int_0^\infty R\, J_0(\mu R)\, f(R)\, dR$$

this implies that an infinite vibrating plate with amplitude function f(R) would produce the desired radiation pattern $g(\vartheta)$. We may invert (3a) by the Fourier-Bessel Theorem, which states that

$$(5) \qquad g(\mu) = \int_{R=0}^{\infty} R\, J_0(R\mu)\, dR \int_{\xi=0}^{\infty} \xi\, g(\xi)\, J_0(R\xi)\, d\xi$$

Hence, from (3a), since $g(\xi) = 0$ for $\xi > k$ we set

$$(6) \qquad f(R) = \int_{\xi=0}^{k} g(\xi)\, J_0(\xi R)\, \xi\, d\xi$$

for all R from 0 to ∞.

The function given in (6) seems to represent a vibrating plate of infinite radius, but in practice we shall use a plate of finite radius A. Hence instead of achieving the desired radiation pattern of (3a), we shall find actually,

$$(3b) \qquad g_A(\mu) = \int_0^A R\, J_0(\mu R)\, f(R)\, dR$$

The difference between (3a), the desired radiation pattern, and (3b), the actual radiation pattern, becomes negligible for large plates by the convergence of the integral in (3a). The closeness of the approximation to a desired radiation pattern, achieved by setting the amplitude of vibration equal to f(R) only over the disk of radius A, will then be limited by the permissible magnitudes of A and k.

EXAMPLES

a) Suppose the following radiation pattern is desired:

$$\begin{cases} \gamma(\vartheta) = 1 & \text{for} \quad 0 < \vartheta < \vartheta_0 \\ \gamma(\vartheta) = 0 & \text{for} \quad \vartheta_0 < \vartheta < \frac{\pi}{2} \end{cases}$$

that is,

$$\begin{cases} g(\mu) = 1 & \text{for} \quad 0 < \mu < \mu_0 \\ g(\mu) = 0 & \text{for} \quad \mu_0 < \mu \end{cases}$$

Then, from (6)

$$f(R) = \mu_0^2 \frac{J_1(\mu_0 R)}{\mu_0 R}$$

and the way in which this function applies to the preceding discussion is obvious.

b) Since the Fourier Bessel Transform of $e^{-\frac{h\mu^2}{2}}$ is $\frac{1}{h} e^{-\frac{R^2}{2h}}$, they will be proportional to $g(\mu)$ and $f(R)$ respectively. It is clear that if $f(R)$ is sensibly different from zero only over a very small range of R, the resulting beam will be very wide; on the other hand the beam can be sharpened to any desired extent by increasing the range of R over which $f(R)$ is appreciably different from zero -- which would involve increasing the radius of the disk accordingly. The "reciprocity" of this pattern has been noticed in the report previously referred to as O.S.R.D. No. 706

APPLICATIONS

In most practical cases the construction of the vibrating plate is not such as to permit any arbitrary continuous variation of the complex amplitude of vibration over the surface of the plate.

For the disk, the simplest case is that in which the complex amplitude is held constant over the entire surface. The "amplitude radiation pattern" of the resulting beam was expressed in (4) as

$$(4) \qquad g(k \sin\theta) = v_1 A^2 \frac{J_1(Ak \sin\theta)}{Ak \sin\theta}$$

For purposes of computation we have taken the radius of the disk $A = 3\pi$ cm, $k = 1$ per cm, and we adjust v_1 so that the maximum amplitude of g is unity. The resulting function has been plotted on the accompanying chart as curve a. If we consider as negligible any parts of the beam which have an amplitude of less than one-tenth of the maximum, we see that this vibrating disk yields a beam with a strong central conical portion together with "side-lobes" which extend approximately to an angle of 39° with the axis of the disk. In what follows we shall take as our criterion of a desirable change, one which will decrease the angle of the principal conical portion, and which will eliminate the "side-lobes". It is assumed that the radius of the disk and k are to remain unchanged.

It has been found possible in practice to break the disk up into a central circular portion and concentric annuli, with the complex amplitude of vibration constant over each section, but differing from one section to the next. This involves, in effect, an attempt

to develop an approximation to a desirable "mode of vibration" function f(R).

We consider, now, the case of a disk with a central circle of radius B and a single annulus of outer radius A. If we now set

$$f(R) = \begin{cases} v_1 & \text{for} \quad R < B \\ v_2 & \text{for} \quad B < R < A \end{cases}$$

the integration of (3) yields

$$(7) \qquad g(k \sin\vartheta) = v_2 A^2 \frac{J_1(A k \sin\vartheta)}{A k \sin\vartheta} + (v_1 - v_2) B^2 \frac{J_1(B k \sin\vartheta)}{B k \sin\vartheta}$$

$$+ v_2 A^2 \left[\frac{J_1(A k \sin\vartheta)}{A k \sin\vartheta} + \left(\frac{v_1}{v_2} - 1\right) \frac{B^2}{A^2} \frac{J_1(B k \sin\vartheta)}{B k \sin\vartheta} \right]$$

The parameter v_2 is to be adjusted so that the maximum amplitude of the beam remains unity, as in the previous case. This leaves us with the parameters v_1 and B by means of which we can control the directional properties of the beam.

As an example, we have evaluated g for the case B = 0.546 A, and $v_1 = 2.67\ v_2$. The result is plotted on the chart as curve b. It can be seen that the "side-lobes" have been almost completely eliminated, but that the central portion has become somewhat wider than in curve a. Further consideration indicates that the further narrowing of the central portion of curve b will lead to an increase in the amplitude of the "side-lobes" and that the central beam can never be narrower than that of curve a.

If the disk were to be divided into three or four concentric sections, it seems reasonable to expect that the beam could be

improved even more, because of the increase in the number of parameters. However, it is not readily apparent just how to vary these parameters to achieve the best results, and therefore a good deal but not an excessive amount of computational work might be involved in this procedure. Thus the expression for a disk with two annuli, with C the innermost radius, B the intermediate radius and A the outermost radius, and with

$$f(R) = \begin{cases} v_1 \text{ for } & R < C \\ v_2 \text{ for } & C < R < B \\ v_3 \text{ for } & B < R < A \end{cases}$$

leads to the following value for g:

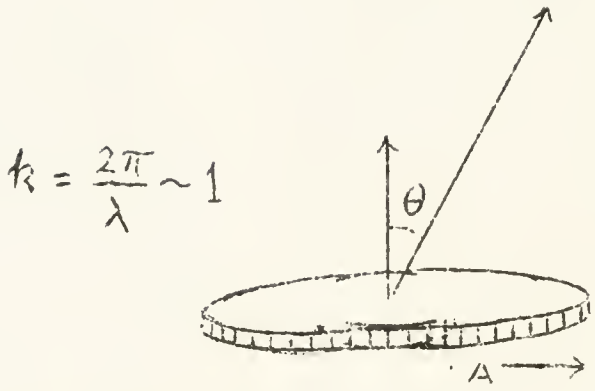
$$(8)\quad g(k \sin\vartheta) = v_3 A^2 \frac{J_1(Ak \sin\theta)}{Ak \sin\theta} + \left(\frac{v_2}{v_3} - 1\right)\frac{B^2}{A^2}\frac{J(Bk \sin\theta)}{Bk \sin\theta} + \left(\frac{v_1}{v_3} - \frac{v_2}{v_3}\right)\frac{C^2}{A^2}\frac{J_1(Ck \sin\vartheta)}{Ck \sin\vartheta}$$

Again v_3 is adjusted to make the maximum amplitude of the beam unity, leaving v_1, v_2, C, and B as parameters to determine the directional properties of the beam. We have made a rough attempt to improve the beam in this case, keeping the v_i real, however. The result is plotted as curve c on the chart, with the values of the parameters indicated there. As can be seen, it is an improvement over both of the other curves. Presumably a better result could be achieved with more care, but it serves to indicate the advantages of this method.

The same method is equally applicable to vibrating plates with other shapes, such as rectangular plates. In many cases it is possible to divide the plate up into a "mosaic" of small sections, each vibrating with its own complex amplitude, so adjusted as to yield a beam with the particular qualities which may be desired. This sort of problem will yield to the kind of analysis indicated above. The smaller the sections, the closer it is possible to approximate to a desirable "mode of vibration" f(R), and hence the closer the approximation to the desired radiational beam.

It is obvious that this method would yield considerably better approximations if the specified size and frequency of vibration of the plate are increased.

Data for Accompanying Chart:

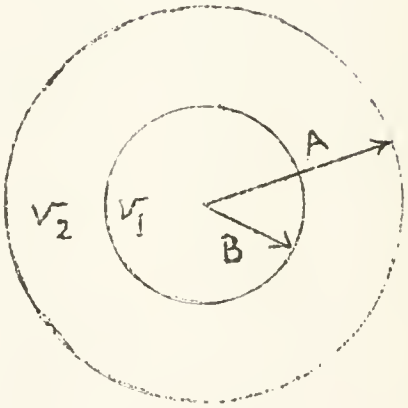


Curve a: For one-section disc or disc with constant amplitude of vibration.

Curve b: For two-section disc:

$B = 0.546\ A$

$V_1 = 2.67\ v_2$



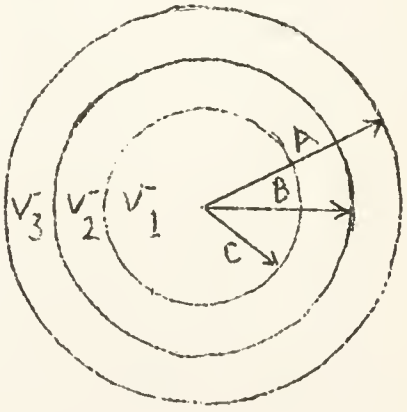
Curve c: For three-section disc:

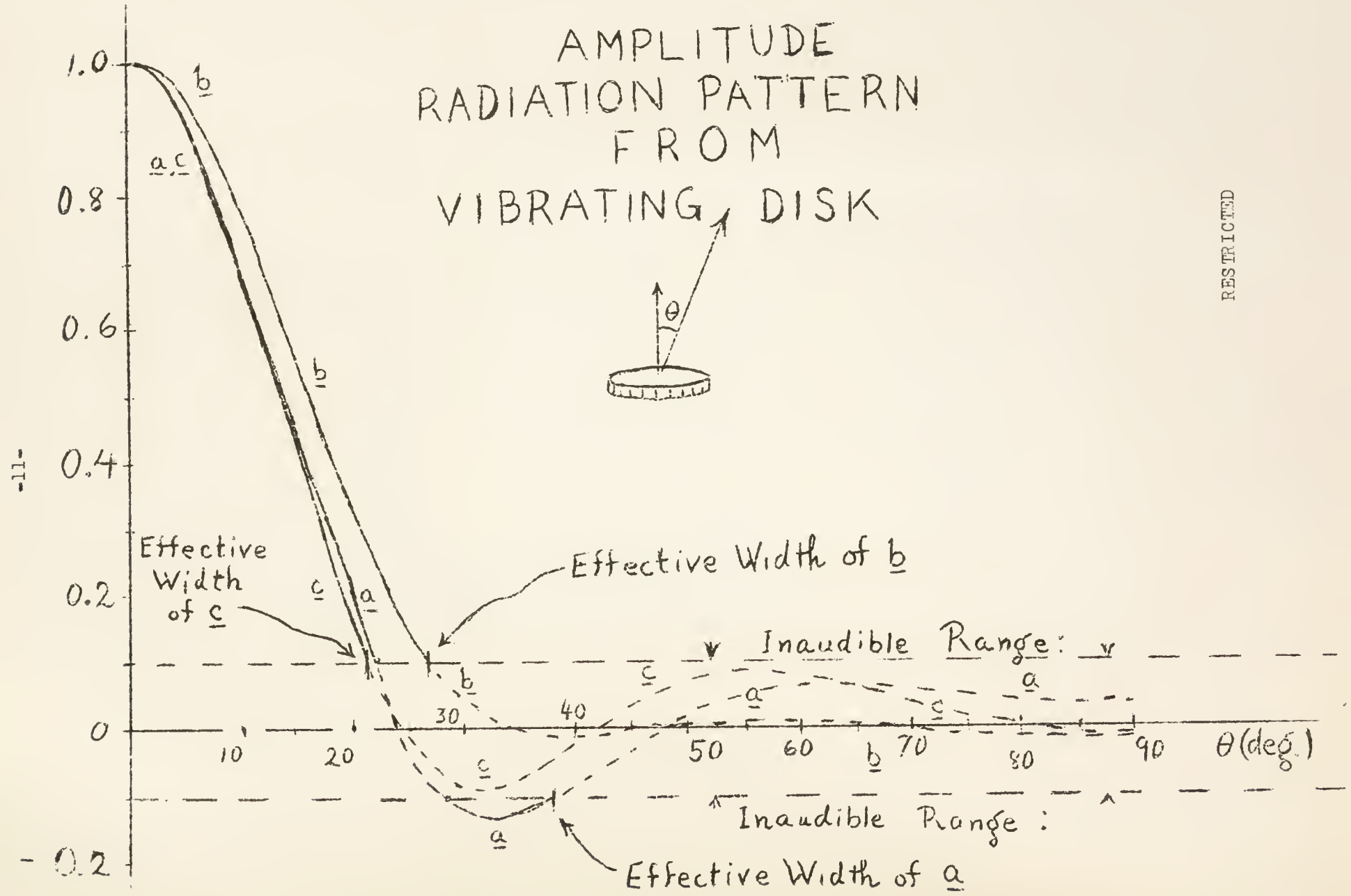
$C = 0.6\ A$

$B = 0.8\ A$

$v_1 = 0.82\ v_3$

$v_2 = 0.22\ v_3$

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